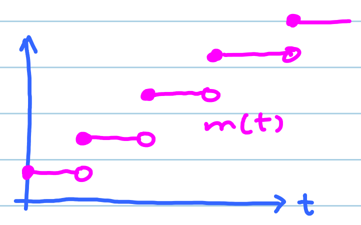
7.2 BW of $x_{FM}(t)$

Recall that the instantaneous frequency of $x_{FM}(t)$ is given by

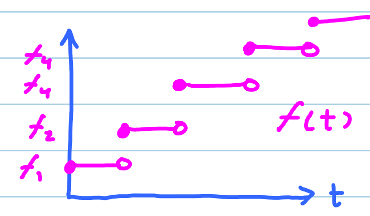
$$f(t) = f_c + k\,m(t) \qquad \star$$

The BW analysis of FM is quite difficult and therefore we will consider a simplified version of $x_{FM}(t)$.

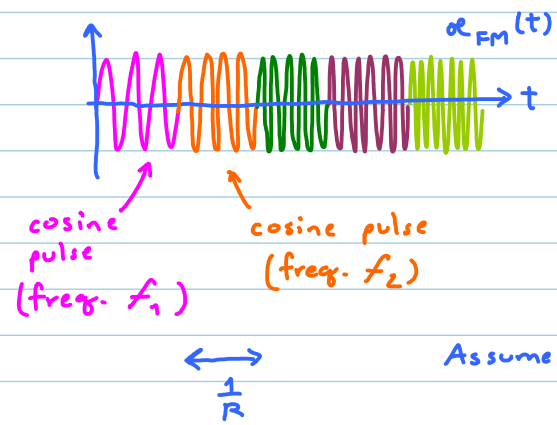
In this simplified version, $m(t)$ is piecewise constant:



By $\star$ above, this also means $f(t)$ is piecewise constant



In class, we demostated this by considering $x_{FM}(t)$ constructed from 5 different tones.



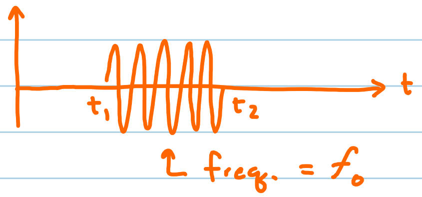
Assume that each tone lasts $\frac{1}{R}$ sec.

Increasing the value of R means we take less time to complete the transmision.

Note that this $x_{FM}(t)$ is a sum of five cosine pulses.

We have seen in the HW that even though the Fourier transform of the cosine function gives two delta functions. The Fourier transform of a cosine pulse gives two sinc functions.

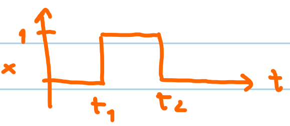
To see this, note that a cosine pulse is simply a product between cosine and rectangular pulse:



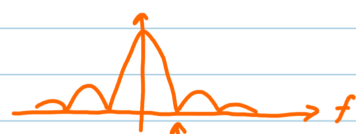
$$= \begin{cases} \cos(2\pi f_0 t), & t_1 \le t < t_2 \\ 0, & \text{otherwise.} \end{cases}$$

$$= \cos(2\pi f_0 t) \times \begin{cases} 1, & t_1 \le t < t_2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \cos(2\pi f_0 t) \times$$



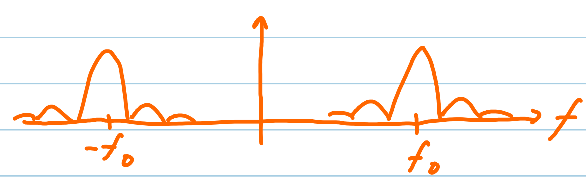
The Fourier transform of the rectangular pulse centered around $t=0$ is a sinc function. Here, the rectangular pulse is shifted to $\frac{t_2+t_1}{2}$ and therefore there will be an extra factor of $e^{-j2\pi f \frac{t_1+t_2}{2}}$ by the time-shift property. This does not change the magnitude of the sinc function. It is still



$$\frac{1}{t_2 - t_1} = \frac{1}{\left(\frac{1}{R}\right)} = R$$

Recall that multiplication by $\cos(2\pi f_0 t)$ will shift the spectrum of the rectangular pulse to $\pm f_0$.

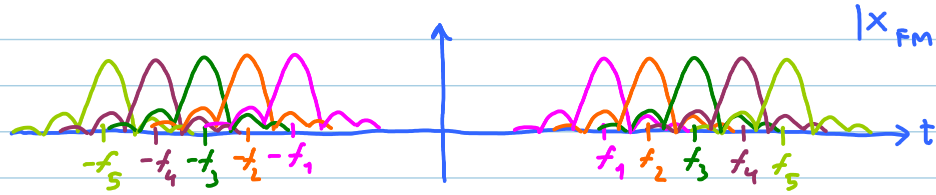
For the cosine pulse, this means its Fourier transform will be two sinc function at $\pm f_0$.



Because $x_{FM}(t)$ is a sum of cosine pulses, we can then see that its spectrum will be a sum of sinc functions centered at $\pm$ frequencies of the pulses

The width of the sinc function is controlled by the time interval each tone takes in the time domain.

Large $R \Rightarrow$ smaller $\frac{1}{R}$ $\Rightarrow$ large mainlope of the sinc function.



$|X_{FM}(f)|$ is the "combination" of these sinc functions.

From the sketch above, we can see that the freq. content of $X_{FM}(f)$ extends to $\pm\infty$. In other words, it is not band-limited.

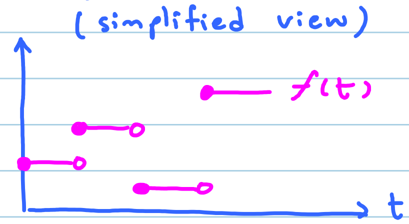
However, one may say that the power of the sinc is mostly contained in its mainlope. In which case, the "BW" becomes approximately

$$R + (f_5 - f_1) + R = (f_5 - f_1) + 2R$$

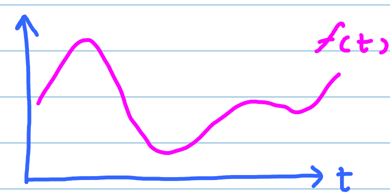
What we have been considering can be regarded as the digitized (or digital) version of FM.

The analog version is the one where $m(t)$ and $f(t)$ are not limited by the piecewise contant behavior.

Digital/ digitized version (simplified view)

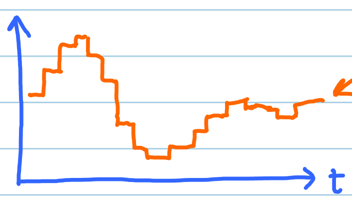


Analog version



This is where the analysis gets more difficult.

Our approach is to try to approximate the analog $f(t)$ above by a piecewise constant function.



If the approximation is good, then we may be able to approximate the BW of $x_{FM}(t)$ from the "modified $x_{FM}(t)$" which is constructed via the approximation.

The value for each piece is determined by the value of $f(t)$ around that time.

$\Rightarrow$ Sampling.